Extra Problems

1. For the function $f(x)$ graphed below, find the following limits or explain why they do not exist.

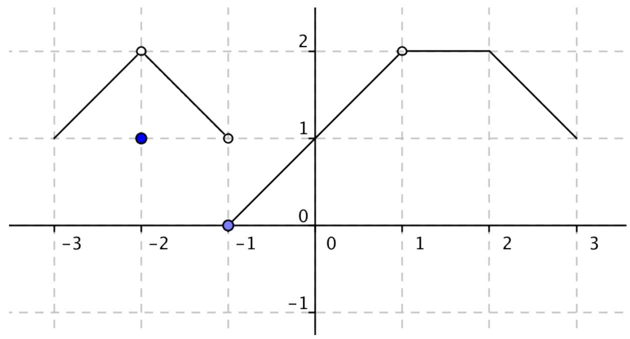


Figure 1: $y = f(x)$

(a) $\lim_{x\to -2} f(x)$

(b) $\lim_{x\to -1} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to 1} f(x)$

Determine if $f(x)$ is continuous at $x = -2$, $x = -1$, and $x = 0$. If it is not, then what type of discontinuity do you find there?

(a) at $x = -2$ (b) at $x = -1$ (c) at $x = 0$

2. Suppose that $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} g(x) = -4$. Find the following limits, in case they exist. Otherwise, briefly explain why the limit does not exist. Show all your work and mention any Limit Laws you use.

(a) $\lim_{x\to 1} \dfrac{2f(x)}{g(x)}$

(b) $\lim_{x\to 1} \sqrt{-g(x)}$

3. Find each limit. If the limit does not exist, explain why. Justify all your answers and show all your work.

(a) $\lim_{x\to 1} \dfrac{x-1}{x^2-1}$

(b) $\lim_{x\to 1} \dfrac{x^2-1}{(x-1)^3}$

(c) $\lim_{x\to \infty} \dfrac{3x^2-5}{5x^2-3}$

(d) $\lim_{x\to 0} \dfrac{\tan 5x}{3x}$

(e) $\lim_{x\to \infty} x \sin \dfrac{1}{x}$

4. Use the Sandwich Theorem to show that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$.

5. Sketch the graph of a function $y = f(x)$ satisfying the following conditions. You do not need to provide a formula for your function, just make sure you identify the important features of the graph.

- $\lim_{x\to -\infty} f(x) = 1$
- $f(-1) = 0$
- $\lim_{x\to 0^-} f(x) = -\infty$
- $\lim_{x\to 0^+} f(x) = \infty$
- $f(2) = 1$
- $\lim_{x\to \infty} f(x) = 0$

6. For what value of a is $g(x) = \begin{cases} x^2+2, & x \le 1 \\ 5ax, & x > 1 \end{cases}$ continuous at every x? Write down all the steps in your solution.

7. Find the continuous extension of $f(x) = \dfrac{x^3 - x^2 + 2x - 2}{x-1}$ to $x = 1$. Call this continuous extension $F(x)$. Write down all the steps in your solution.

8. Let $f(x) = x^3 + 2x - 1$, and note that $f(x)$ is continuous everywhere. Show that the graph of f crosses the $x-$axis between $x = -1$ and $x = 1$. That is, show that the equation $f(x) = 0$ has a solution on the interval $[-1, 1]$. Justify your answer and state any theorems used.

9. Find the slope of the tangent line to $f(x) = \sqrt{x+1}$ at $x = 8$ by using the limit definition of the derivative. Show all your work.

10. Below is the graph of a function $y = f(x)$. The domain of f is all real numbers. Sketch the graph of the derivative of f.

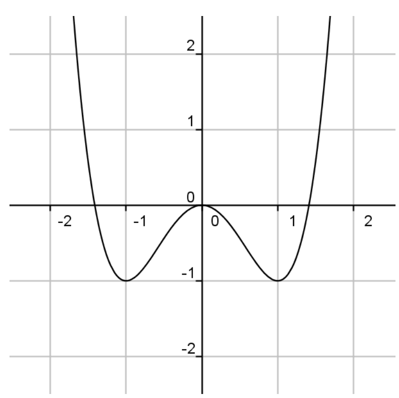


Figure 2: $y = f(x)$

11. Recall that if a function is differentiable, then it is continuous. Give an example of a function that is continuous but not differentiable at $x = 0$. (You can sketch a graph or provide an explicit formula for your function.) Why is your function not differentiable at $x = 0$?

12. Suppose that f and g are differentiable functions of x. Given the following table

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	4	5	2	-2
2	5	-3	-1	-6

find the values of the following derivatives at $x = 1$. Write down every step and state the differentiation rules you use.

(a) $\frac{d}{dx}(f \circ g)$ (b) $\frac{d}{dx}(fg)$ (c) $\frac{d}{dx}\left(\sqrt{f}\right)$

13. Find the derivative of y with respect to x. Show all your work and simplify your answers as much as possible.

    (a) $y = \sqrt{x^2 + 1}$
    (b) $y = \tan(\sec x)$
    (c) $x = \tan y$
    (d) $y = \dfrac{\sin x}{\cos x + 1}$
    (e) $y = \displaystyle\int_1^x \sqrt{t^2 + 1}\, dt$

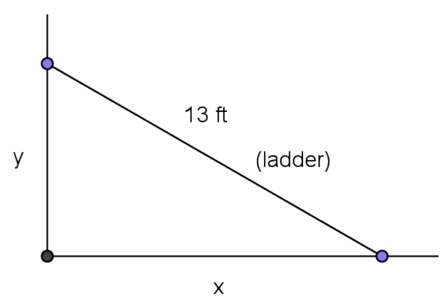
14. Find the derivative of $y = x^2 \sin x + 2x \cos x - 2 \sin x$. Show your work and simplify your answer as much as possible.

15. Find the equation of the tangent line to $y = \dfrac{2x + 5}{3x - 2}$ at the point $(1, 7)$. Write down every step.

16. Use implicit differentiation to find the equation of the normal line to $x^2 + y^2 = 25$ at $(-3, -4)$. Show all your work.

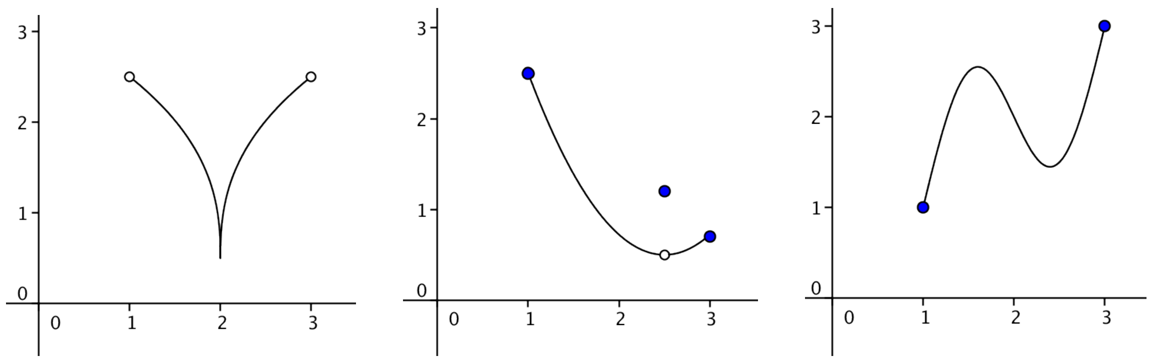
17. Find the values of m and b that make the function $f(x) = \begin{cases} \sin x, & x < \pi \\ mx + b, & x \geq \pi \end{cases}$ continuous and differentiable at $x = \pi$. Write down all the steps in your solution.

18. A 13 ft ladder is leaning against a wall when its base starts to slide away. The base of the ladder is moving at a rate of 5 ft/s when the base is 12 ft from the wall. How fast is the top of the ladder sliding down the wall at that moment? Show all your work.



19. Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 3x$ on $[0, 2]$. Where are these values attained? Show all your work.

20. Consider the following three graphs. Determine if the functions they represent attain an absolute maximum value <u>and</u> an absolute minimum value on $[1, 3]$. If a function does attain its extreme values on $[1, 3]$, find them. Otherwise, explain why the hypothesis of the Extreme Value Theorem is violated.



(a) $y = f(x)$ (b) $y = g(x)$ (c) $y = h(x)$

21. Find $f(x)$ if $f'(x) = 2x - 1$ and the graph of f passes through the point $(0, 1)$. Write down every step in your solution.

22. A driver handed a ticket at a toll booth showing that in 2 hours he had covered 150 miles on a toll road with speed limit 65 mph. The driver was cited for speeding. (That is, at some point, the driver's speed must have exceeded 65 mph.) Explain why. Mention any theorems used.

23. Let $g(x) = 4x^3 - 3x$.

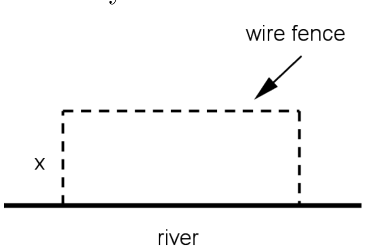
    (a) Find $g'(x)$ and $g''(x)$.

    (b) Use g' and the First Derivative Test for Monotonic Functions to determine where g is increasing/decreasing. Use g' and the First Derivative Test for Local Extrema to determine whether g has any local maxima/local minima.

    (c) Use g'' and the Second Derivative Test for Concavity to determine where g is concave up/concave down. Are there any inflection points?

    (d) Sketch the graph of $y = g(x)$ using the information obtained above.

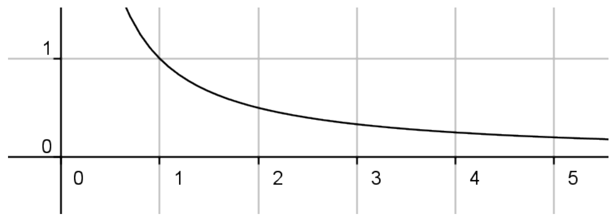
24. A rectangular plot of land will be bounded on one side by a river and on the other three sides by a wire fence. With 800 meters of wire available, what is the largest area you can enclose? Show all your work.



25. A particle moves with acceleration $a(t) = 2/t^3$. Assuming that its velocity at time $t = 1$ is $v(1) = 0$ and its position at time $t = 1$ is $s(1) = 2$, find the position function $s(t)$. Show all your work.

26. Estimating Area

(a) Below is part of the graph of $f(x) = 1/x$. Divide the interval $[1, 5]$ into four subintervals of equal width, and draw the four rectangles associated with the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x = f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x$, where c_k is the left-endpoint of the kth subinterval.



(b) What is Δx? What is $f(c_1)$? $f(c_2)$? $f(c_3)$? $f(c_4)$?

$\Delta x =$ $\quad$ $f(c_1) =$ $\quad$ $f(c_2) =$ $\quad$ $f(c_3) =$ $\quad$ $f(c_4) =$

(c) Now estimate the area under the curve $y = 1/x$ between $x = 1$ and $x = 5$ by evaluating $\sum_{k=1}^{4} f(c_k)\Delta x$.

27. Suppose that you are driving with a friend along a dirt road in a car whose speedometer works but whose mileage counter is broken. To find out how long this road is, your friend records the car's velocity at 10-sec intervals. Estimate the length of the road using right-endpoints. Show all your work.

time (*sec*)	velocity (*feet/sec*)
0	0
10	42
20	24
30	32
40	40
50	30
60	42

28. Suppose that

$$\int_0^2 f(x)\,dx = 3, \quad \int_1^2 f(x)\,dx = 7, \quad \text{and} \quad \int_1^2 g(x)\,dx = -5.$$

Use the properties of the definite integral to evaluate the following integrals. Justify all your steps.

(a) $\int_0^1 f(x)\,dx$

(b) $\int_2^1 g(x)\,dx$

(c) $\int_1^2 [2f(x) + 3g(x)]\,dx$

(d) $\int_1^1 [f(x) + g(x)]\,dx$

29. Use the inequality $\sin x \leq x$, which holds for $x \geq 0$, to find an upper bound for the value of $\displaystyle\int_0^1 \sin x \; dx$. Show all your work.

30. Consider the function $F(x) = \displaystyle\int_1^x \frac{1}{1+t^2} \; dt$. Answer the following questions, and don't forget to justify your steps.

   (a) What is the value of the function F at $x = 1$?

   (b) What is the slope of the tangent to F at $x = 1$?

31. Find the equation of the tangent line to $f(x) = 5 + \displaystyle\int_0^x \frac{1}{\sqrt{t^2+1}} dt$ at $x = 0$. Write down every step.

32. Evaluate the following *indefinite* integrals. Show all your work.

   (a) $\displaystyle\int \sec x(\sec x + \tan x) \; dx$

   (b) $\displaystyle\int \frac{(x^4+1)^2}{x^3} \; dx$

   (c) $\displaystyle\int \frac{x^3}{(x^4+1)^2} \; dx$

   (d) $\displaystyle\int \frac{4}{\sqrt{4x+1}} \; dx$

33. Evaluate the following *definite* integrals. Show all your work. [Hint: For the second one, graph the integrand and use the connection between area and the definite integral.]

   (a) $\displaystyle\int_0^3 (9 - t^2) \; dt$

   (b) $\displaystyle\int_{-3}^0 \sqrt{9-t^2} \; dt$

   (c) $\displaystyle\int_{-3}^0 t\sqrt{9-t^2} \; dt$

   (d) $\displaystyle\int_{-3}^3 t^3\sqrt{9-t^2} \; dt$

34. Find the area of the region bounded by the curve $y = -x^3 + 4x^2 - 3x = -x(x-1)(x-3)$ and the $x-$axis, between $x = 0$ and $x = 3$. (The region is shown below.) Justify your steps.